Exotic Stochastic Processes from Complex Quantum Environments

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Abstract
Stochastic processes are shown to emerge from the time evolution of complex quantum systems. Using parametric, banded random matrix ensembles to describe a quantum chaotic environment, we show that the dynamical evolution of a particle coupled to such environments displays a variety of stochastic behaviors, ranging from turbulent diffusion to Lévy processes and Brownian motion. Dissipation and diffusion emerge naturally in the stochastic interpretation of the dynamics. This approach provides a derivation of a fractional kinetic theory in the classical limit and leads to classical Lévy dynamics.

1. Introduction

The understanding of how stochastic processes emerge from classical dynamical systems is closely related to classical chaos. Often one finds that the dynamics is non–Gaussian, displaying either enhanced or dispersive behavior [1]. One can find such behavior in the interaction of slow and fast degrees of freedom in many–body systems [2], in tracer diffusion in turbulent backgrounds such as the atmosphere, or random potentials, and many more [3]. The wide variety of processes which exhibit anomalous behavior in the transport has led to a variety of theoretical efforts, including fractional extensions of kinetic theory [4–6], random walks in random potentials [7], power law noise

in generalized Langevin equations [8], stochastic webs [4] and Lévy walks and flights [9,3]. While the common thread to these approaches are generalizations of Brownian motion known as Lévy stable laws (discussed below), there is no common theoretical foundation. For instance, fractional kinetic equations are postulated in such a manner as to provide the desired diffusion through scaling arguments.

In turning to the quantum theory, we might ask whether it is possible to realize anomalous diffusion or Lévy stable laws in the time evolution of quantum Hamiltonians [10]. If classical chaos is the origin of the stochastic processes in classical dynamical systems, it is natural to ask whether quantum chaos can be the source of quantum stochastic processes. Starting with the quantum counterpart of classical chaos, namely random matrix theory (RMT), we consider whether the time evolution of such Hamiltonians can generate stochastic processes. We will see that not only is

this possible, but depending on certain properties of the quantum Hamiltonian, one can realize a full range of stochastic processes, including those of Lévy. Further, fractional kinetic theory develops naturally in the semi–classical limit.

To understand diffusion observed in many systems which have $\langle R^2(t) \rangle \sim t^\gamma$, where $\gamma \neq 1$, generalizations of Brownian motion have been sought. Lévy processes provide such an approach. In studies of the extensions of the central limit theorem, P. Lévy found a continuous class of non–Gaussian processes that satisfy the same fundamental equation that gives rise to the theory of Gaussian processes [11]. In one dimension, the Lévy stable laws have the form

$$P(x,t) = \mathcal{L}_\alpha^A(x) = \int \frac{dk}{2\pi} \exp\left\{ikx - A|k|^\alpha\right\} \quad (1)$$

where $0 < \alpha \leq 2$ and $A \propto t$. Gaussian processes correspond to the case where $\alpha = 2$. The Lévy distributions are scale invariant,

$$\mathcal{L}_\alpha^A(x) = A^{-1/\alpha}\mathcal{L}_\alpha^1(xA^{-1/\alpha}), \quad (2)$$

where for $A = 1$ we drop the superscript: $\mathcal{L}_\alpha^1(x) = \mathcal{L}_\alpha(x)$. The scale invariance indicates that the trajectory followed by the random process will not be dominated by one characteristic scale, resulting in a self–similar behavior. With the exception of the Gaussian ($\alpha = 2$), all the Lévy stable laws have infinite second moments [9]. For certain values of α, it is possible to compute the inverse Fourier transform in (1) [12].

2. Quantum Chaotic Environments

We would like to understand the diffusion of a quantum particle interacting with a quantum chaotic environment. Random matrix theory provides a convenient description of quantum chaotic systems, but as it contains no scales or physics, we must develop the notion of a random environment. Since our approach is Hamiltonian based, we envision the particle interacting with a complex quantum system, which might have a non–trivial density of states, and which is inherently chaotic in the sense of RMT fluctuations of the matrix elements. We will assume that the Hamiltonian H_e which describes this 'environment' is time–reversal invariant, so that it is a real, symmetric matrix. (This serves only to simplify the notation). As the background is chaotic, it is not necessarily thermal.

The Hamiltonian for the chaotic environment plus interaction will have the form

$$H_e = h_0(x,p) + h_1(X,x,p). \quad (3)$$

where (x,p) are the coordinates and momenta of the environment, and (X,P) are those for the test particle. h_0 is the Hamiltonian of the environment and h_1 is the interaction with the test particle. It is convenient to choose the fixed basis of h_0 to describe the matrix elements of h_1. In this basis, we denote

$$h_0 \mid n \rangle = \varepsilon_n \mid n \rangle, \qquad (n = 1, ..., N), \quad (4)$$

so that the matrix elements of H_e are

$$[H_e]_{ij} = \varepsilon_i \delta_{ij} + [h_1(X)]_{ij}. \quad (5)$$

There is now control over the density of states of h_0 by suitably choosing the ε_i to describe the system of interest. We would like to introduce a parameter related to the average level density $(\rho(\varepsilon))$ of h_0 as $\beta = 1/T = d\ell n\rho(\varepsilon)/d\varepsilon$, or equivalently, $\rho(\varepsilon) = \rho_0 \exp(\beta\varepsilon)$. We will see that this derivative of the level density naturally plays the role of temperature in the quantum dynamics in certain cases. We now choose h_1 to be chaotic, which allow us to apply the Gaussian orthogonal ensemble (GOE) to the matrix elements. In this case, the chaotic properties of the interaction between the environment and the test particle are built into the correlation function (second cumulant):

$$\langle [h_1(X)]_{ij}[h_1(Y)]_{kl} \rangle = \mathcal{G}_{ij}(X - Y)\Delta_{ijkl}. \quad (6)$$

Here $\Delta_{ijkl} = [\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$, and all other cumulants vanish. A convenient realization of the correlation function is [13–15]:

$$\begin{aligned} \mathcal{G}_{ij}(X) &= \frac{\Gamma^\downarrow}{2\pi\sqrt{\rho(\varepsilon_i)\rho(\varepsilon_j)}} \\ &\times \exp\left[-\frac{(\varepsilon_i - \varepsilon_j)^2}{2\kappa_0^2}\right] G\left(\frac{X}{X_0}\right). \end{aligned} \quad (7)$$

The essential elements of this function are the following. We expect due to selection rules that

h_1 is not a full matrix, but most likely banded in some way. Here the band width is described through κ_0. As the energy increases, the level density typically increases rapidly, so one expects the average interaction matrix elements to reduce, which is accounted for by the $\rho^{-1/2}$ factors. One also expects H_e to decorrelate in X on some length scale X_0 (NB This is not generally the scale on which the eigenvalues of $H_e(X)$ vary as suggested by the avoided level crossings). Finally, $G(x) = G(-x) = G^*(x) \leq 1$, $G(0) = 1$, and the overall strength $\Gamma^\downarrow$ is known as the spreading width. Before we turn to the dynamics, let us consider the statistical nature of h_1.

3. Short Distance Correlations

An interesting consequence of the statistical measure leading to (6) is that the correlation function given by the two–point function must decorrelate no faster than quadratic [16]. In many cases, the essential characteristic of $G(x)$ is governed by its short distance behavior. On short distance scales we approximate:

$$G(x) \approx 1 - c_\alpha \, |x|^\alpha + \dots \tag{8}$$

For systems with smooth parameter dependence, one has $\alpha = 2$. A Dyson process on the other hand has $\alpha = 1$ [16]. Similarly, disordered systems are often modelled with $\alpha = 1$, for instance when one uses $\langle V(x)V(x')\rangle \sim \exp[-|x - x'|]$.

From the requirement that the probability distribution of matrix elements $[h_1(X)]_{ij}$ be always a positive definite function and using the theorems of Bochner, it can be shown that the correlator $G(x)$ has to satisfy the following restriction at short distances

$$0 < \alpha \leq 2. \tag{9}$$

Hence, we obtain a finite range of values α which are allowed for parametric random matrix ensembles.

As the position X of the slow particle changes, the instantaneous energy levels $E_n(X)$ of $[h_1(X)]_{ij}$ change. Using the above expresion for the correlator $G(x)$ one obtains that the average fluctuations are

$$\langle [E_n(X) - E_n(Y)]^2 \rangle = D_\alpha |X - Y|^\alpha. \tag{10}$$

The energy–spacing fluctuations have a behavior, which is similar to a Lévy process characterized by the diffusion constant D_α. The character of these fluctuations in the eigenvalues $E_n(X)$, indicated by α, will be seen to be related to Lévy distributions, which describe the time evolution of the density matrix for a particle evolving in this chaotic bath.

4. Quantum Lévy Processes

We now derive the dynamics of a quantum particle coupled to this quantum chaotic environment. We use the Hamiltonian

$$H_{ij}(X, P) = \delta_{ij} \left[\frac{P^2}{2M} + U(X) \right] + H_{e,ij}(X). \tag{11}$$

We will be interested in the diffusive and dissipative behavior of the test particle induced by the environment, so we will neglect the effects of the potential $U(X)$. The evolution equation for the density matrix of the test particle for this class of RMT Hamiltonians has been derived recently using influence functional techniques [14]. It is given in a high temperature expansion, and up to $o(\beta)$ has the form:

$$
i\hbar \frac{\partial \rho(X, Y, t)}{\partial t} = \left\{ \frac{P_X^2}{2M} - \frac{P_Y^2}{2M} + U(X) - U(Y) \right.
$$

$$
- \frac{\beta \Gamma^\downarrow \hbar}{4 X_0 M} G'\left(\frac{X - Y}{X_0} \right) (P_X - P_Y)
$$

$$
\left. + i\Gamma^\downarrow \left[G\left(\frac{X - Y}{X_0} \right) - 1 \right] \right\} \rho(X, Y, t). \tag{12}
$$

To extract the dynamics analytically, we pass to the weak–coupling limit [14], in which we keep only the leading order term in the correlation function. This corresponds physically to the case where the test particle does not provide significant feedback to the environment. In this limit, we have $G(x) = 1 - |x|^\alpha$ so that $G'(x)$ represents $-\alpha \text{sign}(x)|x|^{\alpha-1}$.

In the absence of a potential $U(X)$, and when the level density is constant in the region of interest ($\beta = 0$), these equations are readily solved for the test particle density matrix $\rho(X, Y, t)$. In

the variables $r = (X + X')/2$, $s = X - X'$, the solution is

$$\rho(r, s, t) = \int dr' \int \frac{dk}{2\pi\hbar} \rho_0 \left(r', s - \frac{kt}{m} \right)$$

$$\times \exp \left[\frac{ik(r - r')}{\hbar} \right.$$

$$\left. + \frac{\Gamma^\downarrow M}{\hbar k} \int_{s-kt/M}^{s} ds' \left[G \left(\frac{s'}{X_0} \right) - 1 \right] \right] \quad (13)$$

The initial density matrix at $t = 0$ is denoted by $\rho(X, Y, 0) = \rho_0(X, Y)$.

The probability of finding the test particle at a position X at a certain time t, denoted $P(X, t)$, is given by the diagonal elements of the density matrix: $P(X, t) = \rho(X, X, t) = \rho(r, s = 0, t)$. This is readily computed from Eq. (20) as:

$$\rho(r, 0, t) = \int \frac{dk}{2\pi\hbar} \exp \left[-k^2 \left[\frac{\sigma^2}{2\hbar^2} + \frac{t^2}{8M\sigma^2} \right] \right.$$

$$\left. - \frac{\Gamma^\downarrow t^{\alpha+1}}{(\alpha+1)\hbar(MX_0)^\alpha} |k|^\alpha + ik\frac{r}{\hbar} \right]. \quad (14)$$

We have assumed that the test particle is initially in a Gaussian wave-packet

$$\psi(X) = \exp[-X^2/4\sigma^2]/[2\pi\sigma^2]^{1/4} \quad (15)$$

$$\rho_0(r, s) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left[-\frac{1}{8\sigma^2}(4r^2 + s^2) \right]. \quad (16)$$

Comparing to Eq. (1), we see that Eq. (21) is a Fourier transform of a Gaussian and a Lévy process, which can be expressed as the convolution of the two distributions,

$$P(X, t) = \int dX' \mathcal{L}_\alpha^{a(t)}(X') \mathcal{L}_2^{b(t)}(X - X'), \quad (17)$$

with

$$a(t) = \frac{\Gamma^\downarrow}{(\alpha+1)\hbar} \left(\frac{\hbar}{MX_0} \right)^\alpha t^{\alpha+1}, \quad (18)$$

$$b(t) = \frac{\sigma^2}{2} + \frac{\hbar^2}{8M^2\sigma^2} t^2. \quad (19)$$

It is interesting to note that these processes are Markovian, having no memory effects which are responsible for the anomalous character. The dynamics of these processes are now labeled by α,

which was shown in Section 3 to be in the range $0 < \alpha \leq 2$. Hence, Eq. (17) shows us that the short distance decorrelations of the adiabatic states of H_e are directly related to the type of stochastic process which develops in the quantum time evolution of the test particle.

5. Turbulent–like Diffusion

For the cases $\alpha < 1$ and $\alpha > 1$, the character of these Lévy processes has been discussed in [10]. When $\alpha = 2$ and $\beta = 0$, $P(X, t)$ is the convolution of two Gaussians, and as a consequence is Gaussian as well. The test particle in this case exhibits turbulent diffusion, with

$$\langle X^2; t \rangle = \int dX\, X^2 \rho(X, X, t) \quad (20)$$

$$= \sigma^2 + \frac{\hbar^2}{4M^2\sigma^2} t^2 + \frac{\Gamma^\downarrow \hbar}{3M^2 X_0^2} t^3 \quad (21)$$

Here one can see that the initial width at $t = 0$ is due to the Gaussian initial condition, and the second term reflects the natural spreading of a wave–packet in the absence of an environment. The t^3 character of the dissipative contribution which arises from the environment is typical of turbulent backgrounds. The range of times under which we expect such diffusion to hold has been discussed in Ref. [18]. In Table 1 we summarize selected properties of the RMT influence functional and its stochastic behaviors.

6. Fractional Kinetic Theory

One of the phenomenological approaches to anomalous diffusion is to postulate a Fokker–Planck (FP) equation which supports the types of diffusion seen in complex classical systems. To this end, a number of extensions of the FP equation have been proposed in which the spatial and/or time derivatives are replaced with derivatives of fractional order. Fractional derivative/integrals are typically defined through the use of integral transforms [20]. In one of the many realizations, for example, one can define $\mathcal{D}_x^\alpha$ to be a derivative (integral) of order α when $\alpha > 0$

$(\alpha < 0)$ as follows. For $\alpha > 0$ let

$$(\mathcal{D}_t^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x dy (x-y)^{\alpha-1} f(y), \quad (22)$$

where $\alpha > 0$ is real, $f(x)$ is an arbitrary function and $\Gamma(\alpha)$ is the Γ–function. For $f(x) = x^\mu$, where $\mu > 0$ is real, this gives

$$\mathcal{D}_x^{-\alpha} x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\alpha)} t^{\mu+\alpha}. \quad (23)$$

Clearly for $\alpha = n$ a positive integer, this is just the n–th iterated integral of $f(x)$. For general $\alpha > 0$ it is a suitable definition for the fractional integral of order α. For negative values of α, let n denote the integer part of $a = -\alpha$. For this range of α we take

$$(\mathcal{D}_x^a f)(x) = \frac{1}{\Gamma(n-a+1)} \quad (24)$$
$$\times \frac{d^{n+1}}{dx^{n+1}} \int_0^x dy \ (x-y)^{n-a} f(y).$$

In this case

$$\mathcal{D}_x^a x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-a)} t^{\mu-a}, \quad (25)$$

which is what one might expect for a derivative of order a. Fractional derivatives can be related to anomalous diffusion by noting that a diffusion process with $\langle x^2 \rangle = Dt^\gamma$ can be derived from a fractional Fokker–Planck equation through the use of scaling arguments [4].

There are many realizations of the fractional FP equations to physical systems, including applications from turbulence to diffusion in viscoelastic of porous media [3]. Generally they are of the form [5,8,4]

$$\mathcal{D}_t^\delta P(Q,t) = \mathcal{D}_{-Q}^\mu (A(Q)P(Q,t)) \quad (26)$$
$$+ \frac{1}{2} \mathcal{D}_{-Q}^\nu (B(Q)P(Q,t))$$

with fractional derivatives in space and/or time. The powers δ, μ, ν are chosen to reproduce anomalous diffusion through scaling formulas such as $Q^2 \sim t^\gamma$, where γ is a function of δ, μ, ν. But there are many limitations at the moment to this type of phenomenological approach. First, one must assume the existence of the coefficients

A, B and so forth, which are now fractional moments. Further, since there is no microscopic underpinning, extensions to higher dimensions become tenuous since many more coefficients are needed, whose origin is then not understood.

By starting with the quantum problem of a test particle in the chaotic quantum environment, we have introduced certain 'microscopic' scales which describe the interactions between the systems. If we now take a semi–classical limit of our influence functional, we will see that its classical analog is a fractional Fokker–Planck equation. Further, the quantum distribution functions are now solutions (in this limit) of that equation.

The fractional Fokker–Planck equation is derived by taking the Wigner transform $f(Q,P,t)$ of the density matrix $\rho(X,Y,t)$

$$f(Q,P,t) = \int \frac{dR}{2\pi\hbar} \exp\left(-\frac{iPR}{\hbar}\right)$$
$$\times \rho\left(Q+\frac{R}{2}, Q-\frac{R}{2}, t\right). \quad (27)$$

The classical transport equation is now defined for $f(Q,P,t)$ by Wigner transforming the evolution equation (19). To leading order in $\hbar$,

$$\frac{\partial f(Q,P,t)}{\partial t} = \int \frac{dR}{2i\pi\hbar^2} \exp\left(-\frac{iPR}{\hbar}\right)$$

$$\left\{ -\frac{\hbar^2}{2M} \partial_Q \partial_R + U\left(Q+\frac{R}{2}\right) - U\left(Q-\frac{R}{2}\right) \right.$$
$$-i\Gamma^\downarrow \left|\frac{R}{X_0}\right|^\alpha + i\gamma\hbar X_0 \alpha \mathrm{sign}(R) \left|\frac{R}{X_0}\right|^{\alpha-1} \partial_R \right\}$$
$$\times \rho\left(Q+\frac{R}{2}, Q-\frac{R}{2}, t\right). \quad (28)$$

We now use the Reisz fractional integro–differential operator which is defined as [20,10]

$$(-\Delta_P)^{\frac{\alpha}{2}} f(P) = \mathcal{F}^{-1} |X|^\alpha \mathcal{F} f(P), \quad (29)$$

where Δ_P is the Laplacian (here with respect to the momentum P), and $\mathcal{F}$ is a Fourier transform from P to X. For our purposes we take $D_P^\alpha = (-i/\hbar)^\alpha (-\Delta_P)^{\alpha/2}$, since $D_P^2[f] = \partial^2 f/\partial P^2$. We also use $\overline{D}_P^\alpha = (-i/\hbar)^\alpha \mathcal{F}^{-1} \mathrm{sign}(X)|X|^\alpha \mathcal{F}$, which has the property $\overline{D}_P^1[Pf] = \partial(Pf)/\partial P$. The

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result is a fractional Fokker–Planck equation in phase space, which is also a fractional extension of Kramers equation:

$$\frac{\partial f(Q,P,t)}{\partial t} + \frac{P}{M}\frac{\partial f(Q,P,t)}{\partial Q} - \frac{\partial U(Q)}{\partial Q}\frac{\partial f(Q,P,t)}{\partial P}$$

$$= \gamma_\alpha \left\{ \overline{D}_P^{\alpha-1}[Pf(Q,P,t)] \right. \tag{30}$$

$$\left. - \frac{2TM}{\alpha\hbar^2}(i\hbar)^\alpha D_P^\alpha[f(Q,P,t)] \right\}.$$

Here $T = 1/\beta$ is the temperature and the operator and the generalized friction coefficient is

$$\gamma_\alpha = \frac{\beta\Gamma^\downarrow\hbar\alpha}{2MX_0^\alpha}. \tag{31}$$

The solutions to this equation follow directly from the Wigner transform of our quantum solutions (20) and (24).

7. Conclusions

The quantum dynamics of test particles in complex quantum environments was studied using influence functional methods. The stochastic character of the dynamical evolution follows from the specific properties of the chaotic/random matrix environment. Markovian dynamics ranging from Lévy to turbulent and to Brownian diffusion are possible. In the classical limit, one can derive a fractional Fokker–Planck equation, which reduces to the well–known Kramer's equation for the special case of a complex environment. This provides a connection between quantum chaos (RMT), stochastic processes and fractional kinetic theory, which can be used to better understand the origins of anomalous diffusion in complex classical and quantum systems.

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Table 1. Selected dynamical limits of the random matrix influence functional of Ref. [15].

Classical Limit	$\alpha = 2$	Kramer's Equation [15]
	$\alpha < 2$	Fractional Fokker–Planck Equation; Lévy Processes [10]
$U(x) = 0$	$\beta = 0,\ \alpha = 2$	Turbulent Diffusion [18]
	$\beta = 0,\ \alpha < 2$	Quantum Lévy Processes [10]
	$\beta \neq 0$, BM limit	Quantum Brownian Motion with $\langle x^2 \rangle = 2Dt$ [15]
	$\beta \neq 0,\ X_0 \sim 1$	$\langle x^2 \rangle = 2Dt$ but non–Maxwellian distributions [15].
Weak Coupling	$\beta \geq 0,\ \alpha = 2$	Caldeira–Leggett Influence Functional [15,19]